Electrical Tech Note — 514

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Resonance

The effect of a resistor in a circuit is not affected by the frequency of the power source for that circuit. If there is a capacitor in the circuit the effect upon the circuit is called capacitive reactance and the value of capacitive reactance is dependant upon the frequency. At a low frequency the capacitive reactance is high for a given value of capacitance and when the frequency is high, the value of capacitive reactance is low. There is an inverse relationship between the capacitive reactance and the frequency. With an inductor in the circuit, there is a direct relationship between inductive reactance and frequency. When the frequency is low for a given value of inductance the inductive reactance is low, and when the frequency is high the inductive reactance for that same inductor is also high. A graph of reactance verses frequency is shown in Figure 514.1. These phenomenon can be used to select and reject particular signal frequencies by a process called filtering. It is not uncommon for unwanted extraneous signals at various frequencies to be present that make it difficult to work with a particular desired frequency of signal. Whether a resistor, capacitor, and inductor are connected in series or in parallel, for a given value of capacitance and inductance, there is one frequency where the inductive reactance and capacitive reactance are equal. At this particular frequency the circuit takes on unique properties that allow the acceptance or rejection of a small band of frequencies. The condition in a circuit containing inductance and capacitance where the inductive reactance and capacitive reactance are equal is called *resonance*. This Tech Note will examine the frequency relationship of capacitors and inductors and how they can be used in electronic circuits to accept or reject certain frequencies and to filter out unwanted signals that make it hard to identify a desired signal. Another common use of a filter is in a direct current power supply where 60 Hz alternating current is converted to direct current. A filter is attached to the rectifier output to smooth out something called the ripple.

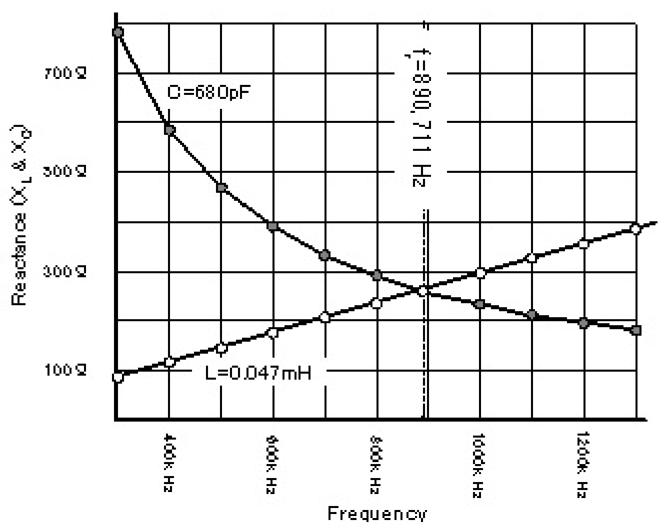

Figure 514.1 *Capacitive reactance decreases as frequency increases while inductive reactance increases as frequency increases.*

Resistance, Reactance, and Impedance: In order to determine the effect of an inductor on a circuit it is necessary to convert the value of inductance into a form that is compatible with Ohm's law since an inductor has a resisting effect on the circuit current. The effect of a specific inductor in a circuit depends upon the frequency of the electrical supply involved. The term used is *reactance* and in this case the quantity is *inductive reactance*. The unit of inductive reactance is the Ohm, but inductive reactance cannot be added arithmetically to resistance. Both are vectors and they act at a right angle to each other. The symbol for reactance is the letter **X** and in particular inductive reactance has a subscript so it is designated as $\mathbf{X_L}$. Equation 514.1 is used to determine the inductive reactance given the value of the inductor in Henry (L) and the frequency of the electrical supply (f) in Hertz. Note by examining Equation 514.1 that when the frequency is 0 Hz, the inductive reactance is zero, and as the frequency becomes very large, the inducive reactance becomes very large. Inductive reactance is the straight line in Figure 514.1 which increases as frequency increases.

$$X_L = 2\pi f L$$ **Equation 514.1**

Now consider a circuit containing a capacitor. In order to determine the effect of the capacitor on a circuit it is necessary to convert the value of capacitance into a form that is compatible with Ohm's law since a capacitor has a resisting effect on the voltage build up across the capacitor. The effect of a specific capacitor in a circuit also depends upon the frequency of the electrical supply involved. The quantity in this case is *capacitive reactance*. The unit of capacitive reactance is the Ohm, and capacitive reactance, similar to inductive reactance, cannot be added arithmetically to resistance. These quantities are vectors and they act at a right angle to each other. The symbol for capacitive reactance is designated as $\mathbf{X_C}$. Equation 514.2 is used to determine the capacitive reactance given the value of the capacitor in Farads (C) and the frequency of the electrical supply (f) in Hertz. Examine Equation 514.2 and note that frequency is in the denominator which means as the frequency approaches zero the capacitive reactance will approach infinity or an open circuit. As the frequency becomes very large, the capacitive reactance approaches zero. Capacitive reactance is the curved line shown in Figure 514.1 decreasing as the frequency increases.

$$X_C = \frac{1}{2\pi f C}$$ **Equation 514.2**

It is important to note that inductive reactance starts at zero when the frequency is 0 Hz and increases to infinity as the frequency becomes very large. On the other hand, capacitive reactance starts at infinity when the frequency is 0 Hz and approaches zero as the frequency becomes very large. This effect is illustrated in Figure 514.1 where reactance in Ohms is plotted on the vertical axis while frequency is plotted on the horizontal axis. Note that there will be some unique frequency where the inductive reactance (X_L) and the capacitive reactance (X_C) will be identical. At this frequency the circuit is said to be in resonance.

When a circuit contains resistance, inductance, and capacitance in series, as shown in Figure 514.2, the net effect of the resistance and reactance is determined using Equation 514.3. For most applications when a capacitor is installed it can be assumed that any resistance associated with the capacitor is insignificant. This means a capacitor represents only capacitive reactance when installed in a circuit. In the case of an inductor, since it is constructed using a long length of wire formed into a coil there is generally a significant amount of resistance associated with the inductor. Generally it must be assumed that resistance and inductance cannot be separated, and when representing an inductor in a circuit it is shown with an inductor symbol in series with a resistor symbol. Inside the inductor these two quantities cannot be separated. Where the phase angle of the capacitor can be assumed to be 90°, the phase angle of an inductor will be less than 90° and can be determined by taking the inverse tangent of the ratio of the inductive reactance divided by the resistance of the inductor. The magnitude of the impedance of the inductor can be determined using Equation 514.3.

$$Z = \sqrt{R^2 + (X_C - X_L)^2}$$ **Equation 514.3**

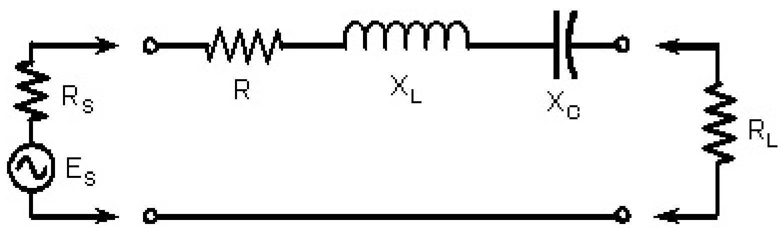

Figure 514.2 *With resistance, inductance, and capacitance connected in series in a circuit the impedance of the circuit reaches a minimum when the inductive reactance and capacitive reactance are identical which will occur at a unique frequency called the resonant frequency.*

Series Resonance: Consider a circuit where a resistor, an inductor and a capacitor are connected in series as shown in Figure 514.2. The impedance of the circuit can be determined using Equation 514.3. Examination of Figure 514.1 shows that there will be a frequency at which the capacitive reactance and the inductive reactance will be identical. It can be seen that in Equation 514.3 that inductive reactance and capacitive reactance are off-setting quantities and when they are equal the impedance of the circuit is equal to the resistance. The condition in a series circuit where inductive reactance and capacitive reactance are equal is called *resonance*. Note in Figure 514.3 that circuit impedance will decrease as the resonant frequency is approached, decreasing to the value of the resistance in series with the inductor and capacitor in the circuit.

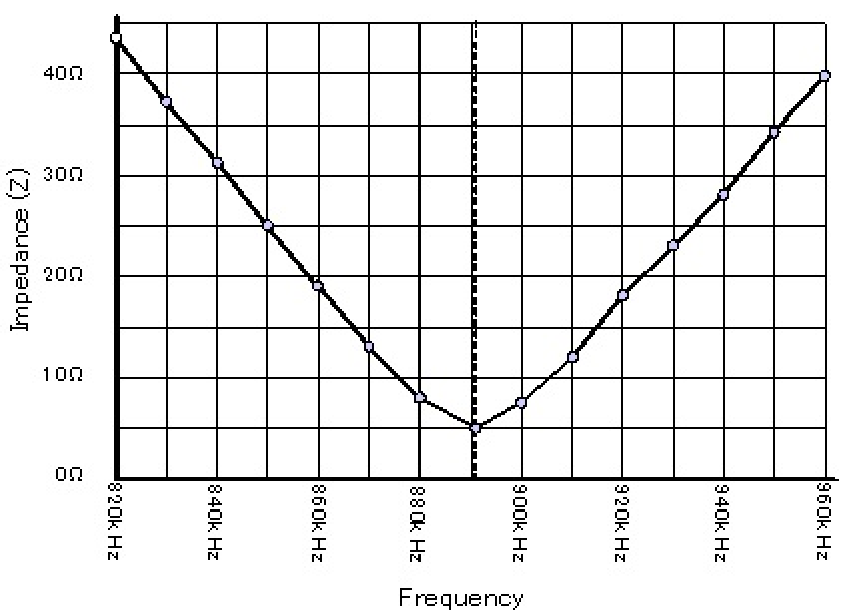

Figure 514.3 *When the value of inductive reactance and capacitive reactance are identical, the impedance of the series circuit will be at a minimum and the current flow will reach a maximum.*

As the frequency increases above the resonant frequency the impedance will increase as illustrated in Figure 514.3. There will be a narrow band of frequencies near the resonant frequency where the circuit impedance is low and the current flow in the circuit will reach a maximum. A current curve is shown in Figure 514.4 reaching a maximum at the resonant frequency. The condition of resonance is also achieved when the inductive reactance equals capacitive reactance for a capacitor and inductor connected in parallel. The parallel resonant circuit will be discussed later as it's effects in a circuit are different than for a series resonant circuit.

If resonance is defined as the frequency where the inductive reactance is equal to the capacitive reactance, then Equation 514.1 can be set equal to Equation 514.2 to determine the resonant frequency for any given value of inductance (L) and capacitance (C). The resonant frequency (f_r) can be determined using Equation 514.4 given the value of inductance and capacitance.

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

Equation 514.4

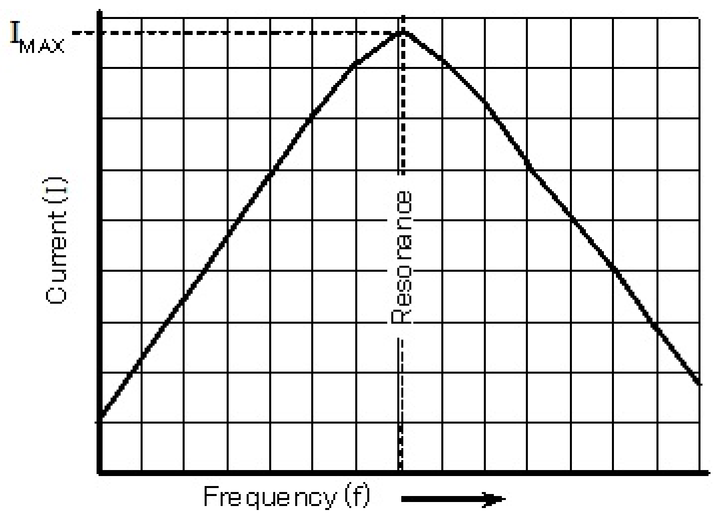

Figure 514.4 *For a series resonant circuit, the current will reach a maximum at the resonant frequency.*

The impedance verses frequency of a series resonant circuit is shown in the graph of Figure 514.3. Note by examining the graph of Figure 514.3 that at lower frequencies the high value of capacitive reactance keeps the impedance of the series circuit high while at the higher frequencies the high value of inductive reactance keeps the circuit impedance high. For frequencies near the resonant frequency there is a distinctive reduction in circuit impedance. The minimum impedance of the circuit occurs at the resonant frequency and is equal to the value of the series resistance. According to Ohm's law, as the impedance of the circuit decreases the current will increase, reaching a maximum at the resonant frequency. Since there is such a great reduction of impedance near the resonant frequency, the effect of frequencies above and below the resonant frequency will be minimized at the output of the circuit. It is this type of resonant circuit that is used to tune a radio to a particular station. Turning the tuner dial changes the value of capacitance in the circuit and thus changes the resonant frequency. Radio stations must transmit at a frequency far enough apart that two or more frequencies are not likely to be within the narrow band of frequencies where the impedance of the circuit is at a minimum.

Quality Factor (Q): Figure 514.5 is a plot of the current of a series resonant circuit with identical values of inductance and capacitance, but with different values of series resistance. Note that even though the frequency at which resonance occurs does not change, the curve for the current verses frequency becomes broader with respect to the maximum current. Note in Figure 514.5 for the current verses frequency that the current reaches a higher maximum when the series resistance is small, and the curve of the current becomes flatter with a lower maximum value when the series resistance is increased. The shape of the curve of current verses frequency is described by a quantity called the *quality* or *quality factor* (Q). By examining Equation 514.3 for impedance of a series circuit, it can be seen that if the resistance is zero the impedance will become zero and the current will become infinitely large at the resonant frequency. This ideal case does not generally exist in real circuits and there will be some minimum value of impedance and some maximum value of current at resonance. The quality factor (Q) is defined as the ratio of the inductive reactance (X_L) of the circuit to the series resistance (R) of the circuit as shown in Equation 514.5. By substituting Equation 514.1 and Equation 514.4 into Equation 514.5, the value of Q can be determined for a series resonant circuit using the values of inductance, capacitance, and resistance. The higher the value of Q for a resonant circuit the higher will be the circuit current in comparison to the current at other frequencies and the narrower will be the band of frequencies at which the current will be a high value.

$$Q = \frac{X_L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$ **Equation 514.5**

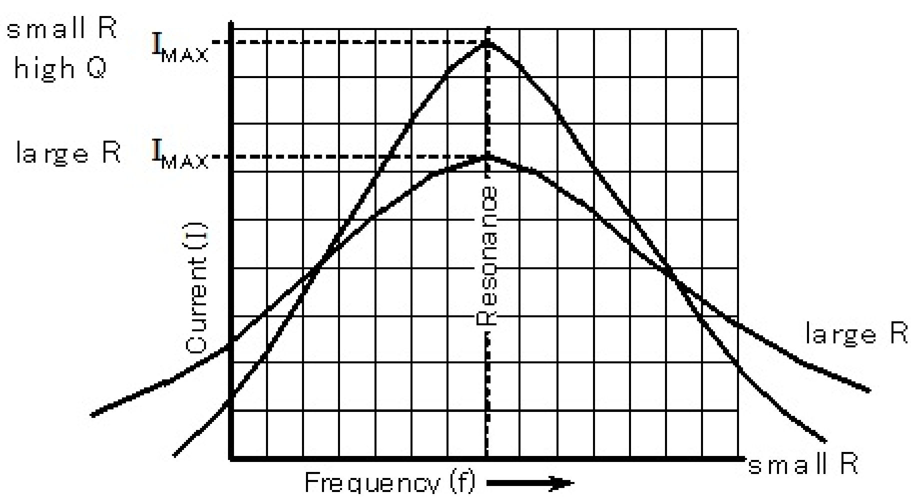

Figure 514.5 *The value of inductance and capacitance will determine the resonant frequency, but the magnitude of the series resistance will determine the shape of the curve of impedance and current verses frequency.*

Band Width: Resonant *bandwidth* for a series resonant circuit is defined as the range of frequencies spanning higher and lower than the resonant frequency for which the power (P) or signal strength is not less than 50% of the power or signal strength at the resonant frequency. Bandwidth is indicated on the current curves of Figure 514.6. Since power (P) is equal to the square of the current times resistance, the points on the curves of Figure 514.6 are where the current has decreased to 70.7% of the value of current at the resonant frequency. Note that the square root of 0.5 is 0.707. When the current has decreased to 70.7% of the value at resonance, the signal strength has decreased to 50%. The frequencies at which the signal strength has decreased to 50% (current decreases to 70.7%) are defined as the upper and lower *cut-off frequencies*. The Q of the circuit has a direct influence upon bandwidth and Equation 514.6 gives the relationship of Q, resonant frequency (f_r) and bandwidth (BW).

$$BW = \frac{f_r}{Q}$$ **Equation 514.6**

The shape of the curves of impedance and current verses frequency are not exactly symmetrical about the resonant frequency. Generally this is not an issue except when precise cut-off points are necessary. It is generally assumed that when the Q for the circuit is the value 10 or greater the curves are for all practical purposes symmetrical about the resonant frequency and the upper and lower cut-off frequencies can be determined by dividing the bandwidth by two and adding to or subtracting from the resonant frequency. Equation 514.7 can be used to determine the upper and lower cut-off frequencies.

$$f_C = f_r \pm \frac{1}{2} BW$$ **Equation 514.7**

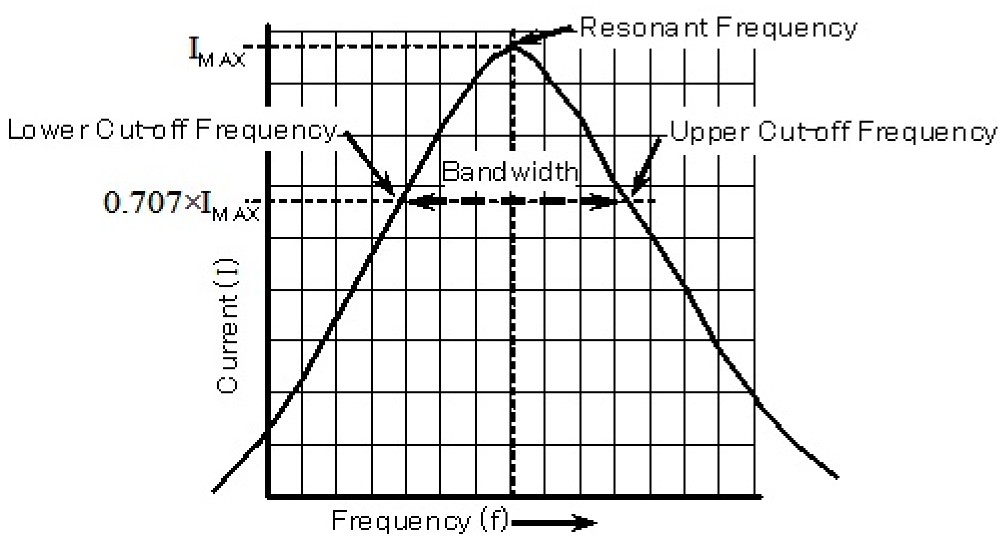

Figure 514.6 *The bandwidth is a range of frequencies about the resonant frequency for which the signal strength or power is not less than 50% of the value at resonance which is the same as the ratio of being not less than 70.7 of the current at the resonant frequency.*

For a series resonant circuit the impedance is at the minimum value at the resonant frequency as shown in Figure 514.4. The impedance increases the farther the frequency deviates from the resonant frequency. In order to determine the upper and lower cut-off frequencies given a curve for impedance, multiply the impedance at resonance by the reciprocal of 0.707 which is actually the square root of two (1.414). This procedure will give the frequencies at which the increase in impedance has caused the power to be reduced by a factor of two.

Parallel Resonant Circuit: Another way an inductor and capacitor can be arranged to form a resonant circuit is for them to be connected in parallel with each other and placed in series with the circuit conductor as shown in Figure 514.7. Note that there is always some resistance in series with the inductance that cannot be discounted. In electronics this arrangement of a capacitor and inductor is called a *tank circuit*. A voltage builds up on the plates of the capacitor and then discharges through the inductor. The energy of the magnetic field of the inductor then induces a current to flow back to the capacitor. If it were not for the small resistance of the tank circuit the electrical energy injected into the tank would continue indefinitely for that frequency at which the capacitive reactance X_C is equal to the inductive reactance X_L. In a circuit where an inductor, a capacitor, and a resistor are connected in parallel, the current in the circuit is determined using Equation 514.8. Note that in Figure 514.7 there is no resistor in parallel with the inductor and capacitor, therefore, the current I_R in Equation 514.8 is zero. When the inductive reactance is equal to the capacitive reactance, the magnitude of the current through the capacitor and inductor will be nearly the same but 180° out-of-phase with each other. In Equation 514.8, the current of the inductor will cancel the current of the inductor and the net current flowing in the circuit will be nearly zero. In order for the net current of the circuit to be zero the impedance must be infinite. This is the unique aspect of a resonant circuit where a capacitor is connected parallel with an inductor. The current of the circuit is very low at the resonant frequency and the circuit impedance is very high.

$$I_{LINE} = \sqrt{I_R^2 + (I_L - I_C)^2}$$ **Equation 514.8**

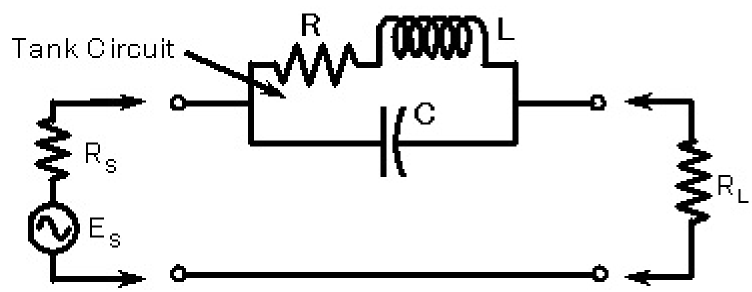

Figure 514.7 *A parallel resonant circuit is formed when a capacitor and an inductor are arranged in parallel and connected into the circuit. There is always some resistance in series with the inductance.*

The current and impedance of a parallel resonant circuit is shown in Figure 514.8. There is always some resistance associated with the inductor, therefore, the current will never be zero and the impedance will be high, but much less than infinity. The parallel resonant circuit can, therefore, be used to reject a particular frequency or a narrow band of frequencies near the resonant frequency.

The resonance for a circuit with a capacitor and an inductor connected in parallel is also defined as the frequency where the inductive reactance (X_L) and capacitive reactance (X_C) are equal. For a parallel resonant circuit the **resonant frequency** (f_r) is also determined using Equation 514.4.

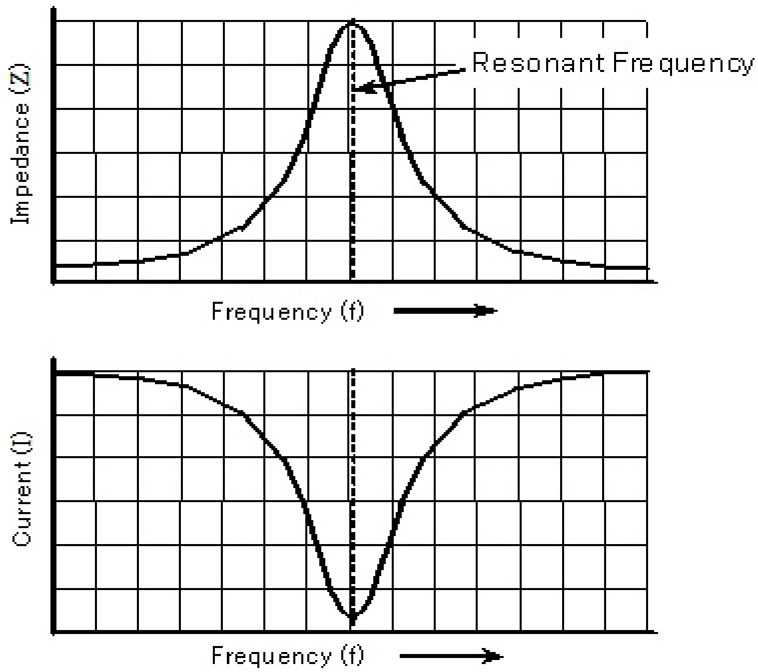

Figure 514.8 *For a parallel resonant circuit the impedance is very high at the resonant frequency resulting in a very low current.*

The **quality factor (Q)** also describes the shape of the impedance and current response curves with respect to change of frequency. As the resistance in series with the inductance increases the quality factor decreases according to Equation 514.5 the same as the case of a series resonant circuit. The upper and lower cut-off frequencies are also defined as the points where the signal strength or power is reduced to 50% which is the point where the impedance of the circuit decreases to 70.7% of the maximum value at the resonant frequency. Since the current has reached a minimum at this point, the **cut-off frequencies** are those points where the current is 1.414 times the minimum value at resonance. As was the case with a series resonant circuit, the **bandwidth** is the range of frequencies between the lower cut-off frequency and the upper cut-off frequency. The bandwidth of a parallel resonant circuit is also determined using Equation 514.6 by dividing the resonant frequency by Q, and the upper and lower cut-off frequencies can be determined using Equation 514.7.

The value of the resistance in series with the inductance obviously has an effect upon the value of Q as can be seen by examining Equation 514.5. Figure 514.9 is a graph of the impedance of a parallel resonant circuit verses frequency showing the effect of the series resistance on the overall impedance of the parallel resonant circuit. The curve with the high Q has a value well in excess of 100 and the curve marked low Q has a value of less then 10. It may be desirable to adjust the value of series resistance in order to achieve rejection of a band of undesirable frequencies. Sometimes a resistor is installed in series with the inductor to achieve the value of Q desired. Another way to increase the bandwidth for a parallel resonant circuit, sometimes called a *wave trap*, is to install a resistor in parallel with the inductor and capacitor. In this case Equation 514.8 will govern the line current of the circuit with a value of I_R always flowing around the wave trap. There are a number of options that can be used by a circuit designer to achieve the desired result of a circuit.

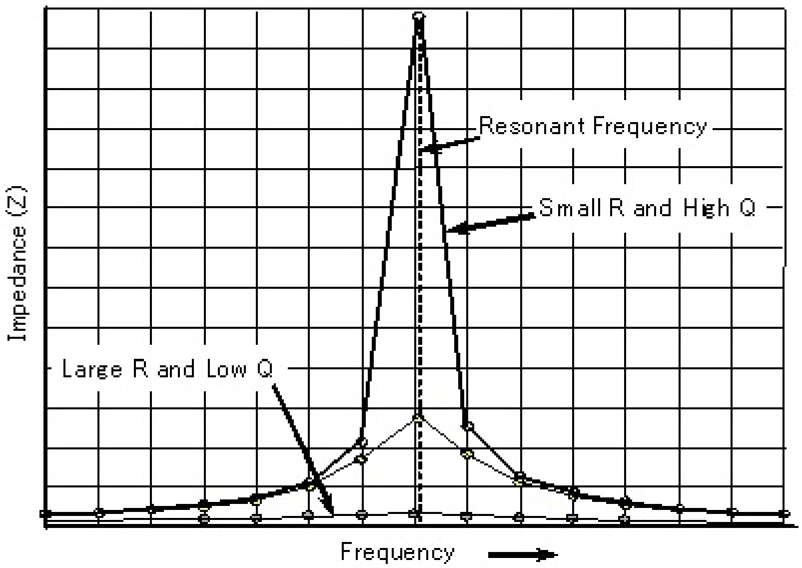

Figure 514.9 *The value of Q, which is influenced by the resistance in series with the inductor, effects the shape of the impedance verses frequency curve with a very high impedance and narrow bandwidth achieved with a high value for Q.*

The parallel resonant circuit shown in Figure 514.7 is referred to as a tank circuit in electronics. In addition to the line current (I_{LINE}) that flows through the circuit there is a separate current that flows in the tank circuit between the capacitor and the inductor (I_{TANK}). At resonance this tank current can get very high when the line current is actually very low. The Q of the parallel resonant circuit is determined using Equation 514.5 by dividing the inductive reactance (X_L) by the resistance (R). This is an accurate way to determine Q, but the value of Q is actually the ratio of the current flowing in the tank (I_{TANK}) to the current flowing in the line (I_{LINE}). Tank current can then be determined using Equation 514.9. The line current is determined by dividing the voltage across the tank circuit by the impedance of the tank. The tank is a parallel circuit and it is a little difficult to calculate the impedance, but there is a simple method of closely estimating the value of tank impedance (Z_{TANK}) by multiplying the inductive reactance (X_L) by the Q as shown in Equation 514.10. Once the line current has been determined Ohm's law is used to determine the line current (I_{LINE}) by dividing the voltage across the tank (E_{TANK}) by the impedance of the tank (Z_{TANK}) as shown in Equation 514.11. The reason this may be an important calculation is that the components in the tank must be capable of carrying the tank current (I_{TANK}). For many circuits where there is a parallel resonant circuit (tank circuit) the currents involved are well below the ratings of the components, but for some applications this tank current can be significant enough to cause damage to the circuit components.

$$I_{TANK} = Q \times I_{LINE}$$

Equation 514.9

$$Z_{TANK} = Q \times X_L$$

Equation 514.10

$$I_{LINE} = \frac{E_{TANK}}{Z_{TANK}}$$

Equation 514.11

Conclusion: Series and parallel resonant circuits have many applications in electronics. A common application is to tune a circuit to accept a particular frequency of incoming signal. Another common application is in filtering circuits, where desired frequencies are permitted to pass through for signal processing and other frequencies are filtered out or rejected. In particular resonant circuits are used in frequency band pass and band reject circuits. Knowing how capacitors and inductors respond to the frequency range is extremely important in electronics.