Electrical Tech Note — 230

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Electrical Circuit Analysis

(Kirchhoff Laws, Superposition, Thevenin & Norton Equivalents)

Introduction: When there are more than one voltage source in a circuit, determination of a desired quantity in the circuit generally cannot be accomplished using circuit reduction. Other techniques are generally necessary such as solving simultaneous equations based upon the Kirchhoff voltage or current law, by circuit superposition, or by analyzing a Thevenin or a Norton equivalent circuit. These circuit analysis techniques will be discussed. A basic understanding of matrix algebra is necessary to solve simultaneous equations, although the examples used are simple enough to solve using simple variable substitution. For a basic review of circuit analysis please review Tech Note 215.

Kirchhoff's Voltage Law: Kirchhoff's voltage law states that the voltages around any circuit loop will add up to zero. This means that moving around a circuit loop and ending back at the starting point will result in returning to the starting voltage. A simple example of a series circuit will illustrate the voltage law. Figure 230.1 is a circuit with two dc voltage sources in series with three resistors. Assuming conventional current flow, the direction of current flow is from the positive (+) terminal, through the circuit to the negative (–). As the current passes through a load such as a resistor, a voltage drop will occur according to Ohm's law. Since the resistor is a power sink rather than a source, the polarity will be opposite to a source. Current will enter the resistor at the positive (+) end of the voltage drop created and will emerge from the negative (–) end of the voltage. Kirchhoff's voltage law is as follows for the circuit of Figure 230.1. First determine the total resistance of the circuit (R_T) and the circuit current (I).

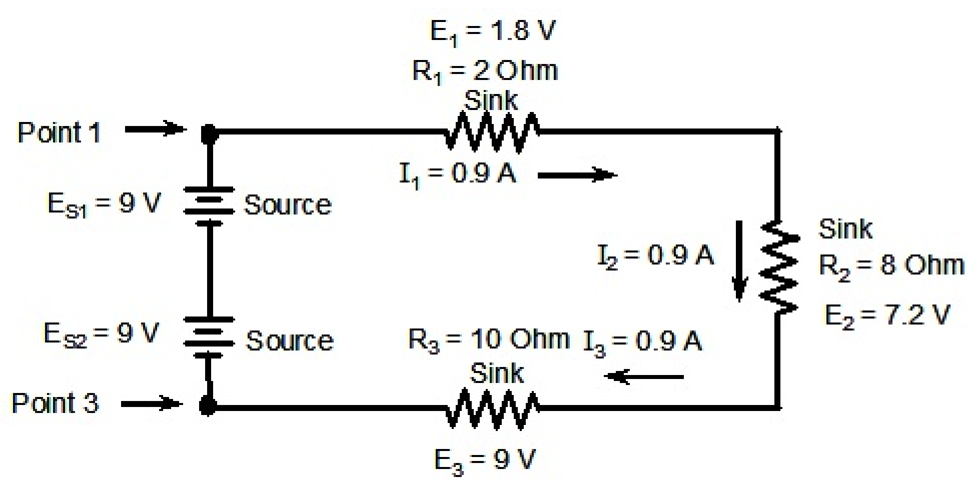

Figure 230.1 *The sum of the source voltages and the sink voltage drops of a series circuit will add up to zero according to Kirchhoff's voltage law.*

$$R_T = R_1 + R_2 + R_3 = 2 \text{ ohm} + 8 \text{ ohm} + 10 \text{ ohm} = 20 \text{ ohm}$$

$$I = \frac{18 \text{ V}}{20 \text{ ohm}} = 0.9 \text{ A}$$

$$E_{S1} + E_{S2} - (R_1 \times I) - (R_2 \times I) - (R_3 \times I) = 0\text{V}$$

$$9 \text{ V} + 9 \text{ V} - (2 \text{ ohm} \times 0.9 \text{ A}) - (8 \text{ ohm} \times 0.9 \text{ A}) - (10 \text{ ohm} \times 0.9 \text{ A}) = 0 \text{ V}$$

$$9\text{V} + 9\text{V} - 1.8\text{V} - 7.2\text{V} - 9\text{V} = 0\text{V}$$

Kirchhoff's Current Law: An additional resistor ($R_4 = 2.5\ \Omega$) was connected between point 2 and point 3 of the previous circuit. The new circuit is shown in Figure 230.2. The current flow through the circuit of Figure 230.2 will be different than the current flow through the circuit of Figure 230.1 because the total resistance (R_T) is different for each circuit. The total resistance (R_T) for the circuit of Figure 230.2 is 12 ohm. First determine the value of a single resistor that can replace the parallel combination of R_3 and R_4. Then add that resistance to the value of R_1 and R_2 to determine the total resistance.

$$\frac{R_3 \times R_4}{R_3 + R_4} = \frac{10 \text{ ohm} \times 2.5 \text{ ohm}}{10 \text{ ohm} + 2.5 \text{ ohm}} = \frac{25 \text{ ohm}}{12.5 \text{ ohm}} = 2 \text{ ohm}$$

$$R_T = 2 \text{ ohm} + 8 \text{ ohm} + 2 \text{ ohm} = 12 \text{ ohm}$$

Next determine the total current for the circuit using Ohm's law.

$$I = \frac{18 \text{ V}}{12 \text{ ohm}} = 1.5 \text{ A}$$

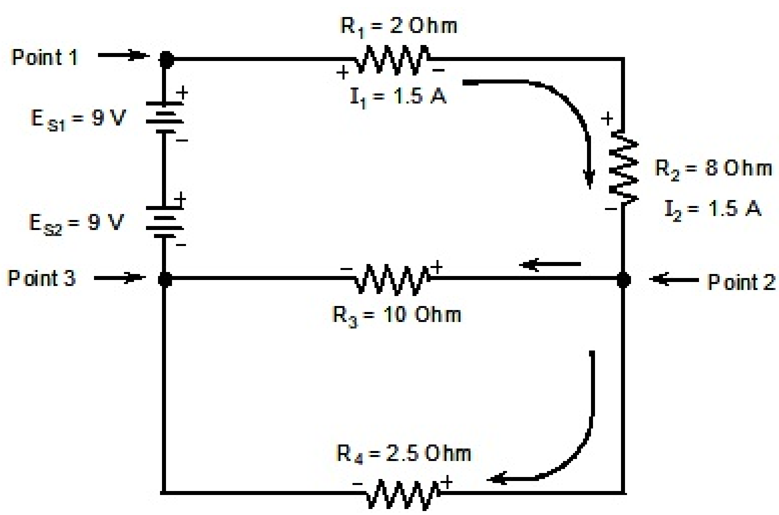

Figure 230.2 *A second resistor (R_4) with a value of 2.5 ohm was connected in parallel with the 10 ohm resistor (R_3) to form a node at point 2 where three branches of the previous circuit are connected.*

This total current passes through R_1 and R_2, but splits up at point 2 with some current flowing through R_3 and some through R_4. Point 2, where several branches of the circuit connect, is called a **node**. According to Kirchhoff's current law, at any node, the current in all of the branches will add up to zero. This law is only logical. The current that flows into a node must flow out of the node. Using Ohm's law, the voltage drop across R_1 and across R_2 can be determined. According to Kirchhoff's voltage law, these two voltage drops can be subtracted from the source voltage to determine the remaining voltage across the parallel resistors R_3 and R_4. These values are shown in Figure 230.3. Now using Ohm's law, determine the current flow through R_3 and R_4. Since their values are not equal, the total circuit current will not divide up equally. There will be more current through the 2.5 ohm resistor than through the 10 ohm resistor.

$$E_1 = R_1 \times I = 2 \text{ ohm} \times 1.5 \text{ A} = 3 \text{ V}$$

$$E_2 = R_2 \times I = 8 \text{ ohm} \times 1.5 \text{ A} = 12 \text{ V}$$

$$E_3 = E_4 = 18 \text{ V} - 3 \text{ V} - 12 \text{ V} = 3 \text{ V}$$

$$I_3 = \frac{3 \text{ V}}{10 \text{ ohm}} = 0.3 \text{ A}$$

$$I_4 = \frac{3 \text{ V}}{2.5 \text{ ohm}} = 1.2 \text{ A}$$

Kirchhoff's current law for Node 2:

$$I_T - I_3 - I_4 = 0 \text{ A}$$

$$1.5 \text{ A} - 0.3 \text{ A} - 1.2 \text{ A} = 0 \text{ A}$$

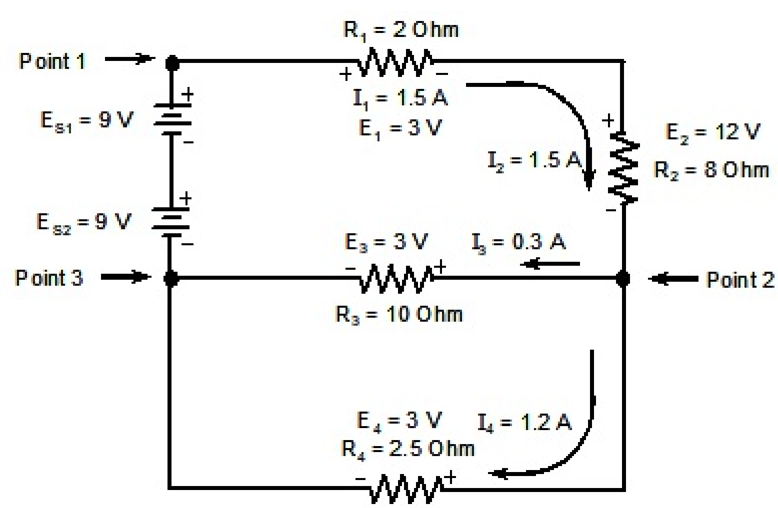

Figure 230.3 *Ohm's law and Kirchhoff's voltage law are used to determine the voltage drop across each resistor and the current through each resistor of the previous circuit.*

Circuit Analysis Using Kirchhoff's Voltage Law: When a circuit contains more than one voltage source located in different branches of the circuit, analysis of the circuit requires techniques other than simple circuit element reduction. Figure 230.4 is a two loop circuit with voltage sources in different elements. This circuit uses ac voltage sources rather than dc sources. The circuit can still be analyzed as a dc circuit provided polarity of the ac sources is known with relation to each other. The ac sources are also assumed in this case to be in phase with each other. Even through the voltage is constantly changing, the rms value of the voltage can be used and a polarity established by freezing time. For Figure 230.4, the two ac voltage sources have a value of 120 volts rms and their polarity is such that the voltages will add. For the following analysis the circuit will be solved using Kirchhoff's voltage law. Start by drawing in a current for each of the loops of the circuit. This circuit has two loops. It is usually most convenient to point the current from the positive to the negative terminals of the voltage source.

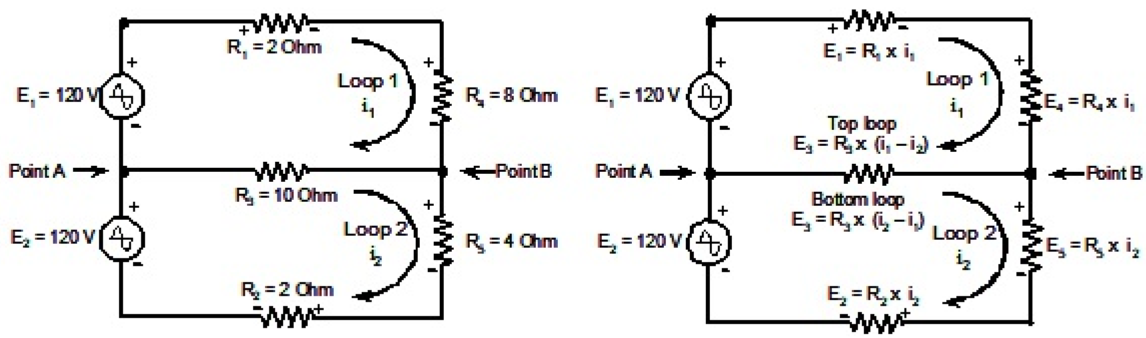

Figure 230.4 *A two loop circuit has two 120 volt ac sources, one in each loop with the polarity such that the two voltage sources will add.*

Next write a voltage equation for each loop with the voltage across each element being the voltage drop across each element. Note that for the resistor common to both loops (R_3), there will be two currents flowing through the resistor. Since there are only two equations for this circuit, it is easy to solve for the two currents (i_1 and i_2) simultaneously. The following solution uses determinants and matrix algebra to solve for the two currents. For circuits with many loops, using matrix algebra is probably the most efficient solution technique.

$$R_1\, i_1 + R_4\, i_1 + R_3(i_1 - i_2) = E_1$$

$$2 i_1 + 8 i_1 + 10(i_1 - i_2) = 120\text{ V}$$

$$2 i_1 + 8 i_1 + 10 i_1 - 10\, i_2 = 120\text{ V}$$

$$20\, i_1 - 10\, i_2 = 120\text{ V}$$

$$R_3(i_2 - i_1) + R_5\, i_2 + R_2\, i_2 = E_2$$

$$10(i_2 - i_1) + 4 i_2 + 2 i_2 = 120\text{ V}$$

$$10 i_2 - 10\, i_1 + 4 i_2 + 2 i_2 = 120\text{ V}$$

$$-10\, i_1 + 16 i_2 = 120\text{ V}$$

$$20\, i_1 - 10\, i_2 = 120\text{ V}$$
$$-10\, i_1 + 16 i_2 = 120\text{ V}$$

$$\begin{vmatrix} 20 & -10 \\ -10 & 16 \end{vmatrix} \bullet \begin{vmatrix} i_1 \\ i_2 \end{vmatrix} = \begin{vmatrix} 120 \\ 120 \end{vmatrix}$$

Cramer's rule can be used to solve for the loop currents (i_1 and i_2).

$$i_1 = \frac{\begin{vmatrix} 120 & -10 \\ 120 & 16 \end{vmatrix}}{\begin{vmatrix} 20 & -10 \\ -10 & 16 \end{vmatrix}} = \frac{1920 - (-1200)}{320 - 100} = \frac{3120}{220} = 14.18A$$

$$i_2 = \frac{\begin{vmatrix} 20 & 120 \\ -10 & 120 \end{vmatrix}}{\begin{vmatrix} 20 & -10 \\ -10 & 16 \end{vmatrix}} = \frac{2400 - (-1200)}{320 - 100} = \frac{3600}{220} = 16.36A$$

Substitute the loop currents (i_1 and i_2) back into the original equation to check the answer.

$20\,i_1 - 10\,i_2 = 120$ V
20×14.18 A $- 10 \times 16.36$ A $= 283.6$ V $- 163.6$ V $= 120$ V

$-10\,i_1 + 16\,i_2 = 120$ V
-10×14.18 A $+ 16 \times 16.36$ A $= -141.8$ V $+ 261.8$ V $= 120$ V

The voltage drop across each component for each loop sum to the source voltage which satisfies Kirchhoff's voltage law. The conclusion is that the solution is correct. The current through each component is shown for the original circuit in Figure 230.5. The currents through resistor R_3 are in opposite directions (180° out of phase) and it is the net current that can be measured.

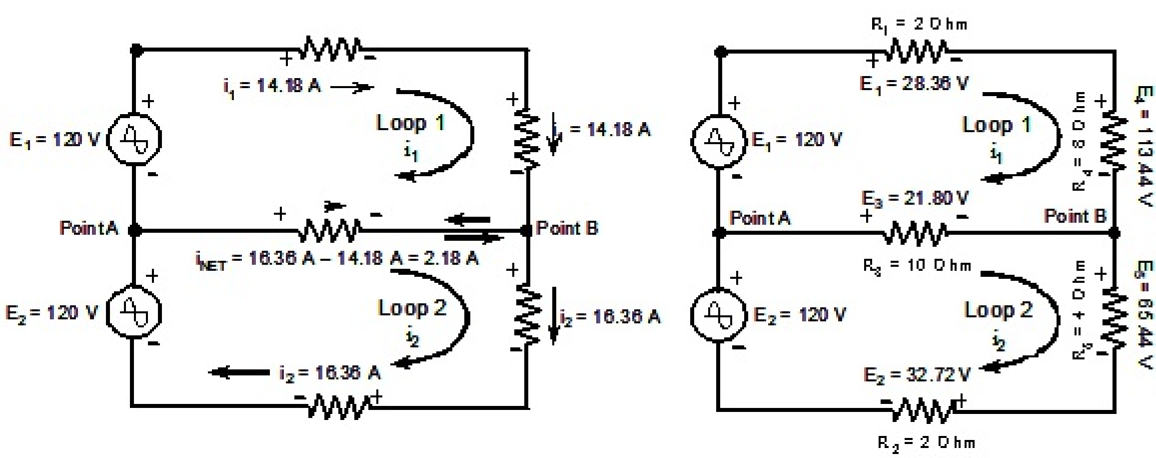

Figure 230.5 *The currents through the components of the circuit of Figure 230.4 are shown as determined using Kirchhoff's voltage law.*

Since this is only a two loop circuit the two loop equations for the circuit can be solved easily as simultaneous equations. Solve the second equation for i_2 and then substitute that solution back into the first equation.

$$16\, i_2 = 120\text{ V} + 10\, i_1$$

$$i_2 = \frac{120\text{ V}}{16} + \frac{10}{16}\, i_1 = 7.5 + 0.625\, i_1$$

$$20\, i_1 - 10\, i_2 = 120\text{ V}$$

$$20\, i_1 - 75 - 6.25\, i_1 = 120\text{ V}$$

$$13.75\, i_1 = 195\text{ V}$$

$$i_1 = \frac{195\text{ V}}{13.75} = 14.18\text{ A}$$

Note the value of i_1 is the same as was determined using the matrix method solution. Next substitute this value for i_1 into the second equation to determine the value of i_2. Note also the value for i_2 is the same as determined using the matrix solution.

$$i_2 = 7.5 + 0.625\, i_1 = 7.5 + 0.625 \times 14.18 = 7.5 + 8.86 = 16.36\text{ A}$$

Figure 530.5 also shows the voltage across each resistor with the current through each resistor multiplied by the value of each resistor. An arrow at each resistor indicates the direction of current through each resistor. For resistor R_3 the arrow indicates the net current flow. Note the polarity of the voltage across each resistor, and add the voltages for each loop. The voltages for loop 2 add to 119.96 volts rather than 120 volts because of round-off error.

Loop 1 of Figure 530.6: +28.36 V +113.44 V – 21.80 V = 120 V

Loop 2 of Figure 530.6: +21.80 +65.44 V +32.72 V = 119.96 V

Kirchhoff's voltage law is satisfied for both loops of this circuit, therefore, the solution for current determined by either of the previous techniques must be correct.

Circuit Analysis Using Superposition: This circuit analysis technique uses basic circuit reduction techniques and does not require the solving of simultaneous equations or a matrix. One advantage of this technique is that it provides an opportunity to study the influence of each voltage source on a circuit. When this technique is used, it is extremely important to identify the direction of current flow through each circuit element for each voltage source. The circuit is analyzed for each voltage source, and then each analysis is superimposed upon each other like layers. The total current flow through each circuit element is the sum of the current flows for each individual voltage source through that element. This superposition technique will be demonstrated by solving the previous example.

Put a shunt around all voltage sources but one and solve the circuit for the remaining voltage source. Make sure the direction of current flow through each circuit element is correct. Determine the current for each component of the circuit caused by each voltage source separately. In the case of the circuit of Figure 230.4, there are two voltage sources and the circuit will be solved two times. Then the two circuits will be superimposed on each other and the net current for each component will be determined.

In Figure 230.6, the voltage source in the lower loop is shorted across and the circuit is solved for the voltage source in the top loop. First the total resistance (R_T) of the circuit is determined so that the total current (I_T) can be calculated using Ohm's law. This circuit reduces to a total resistance (R_T) of 13.75 ohm. The total current (I_T) that flows due to the voltage source in the top loop is 8.73 amperes. This current flows through resistors R_1 and R_4, and splits at the node to flow through the remainder of the circuit. The current through the 10 ohm resistor (R_3) will be 3.27 amperes, and the current through the other branch (R_2 and R_5) which is 6 ohms will be 5.46 amperes. Use any technique desired to determine the current through the parallel branches of the circuit. In this case the current divider formula for two parallel resistors was used. The arrows in Figure 230.6 indicate the correct direction of current flow through each component of the circuit.

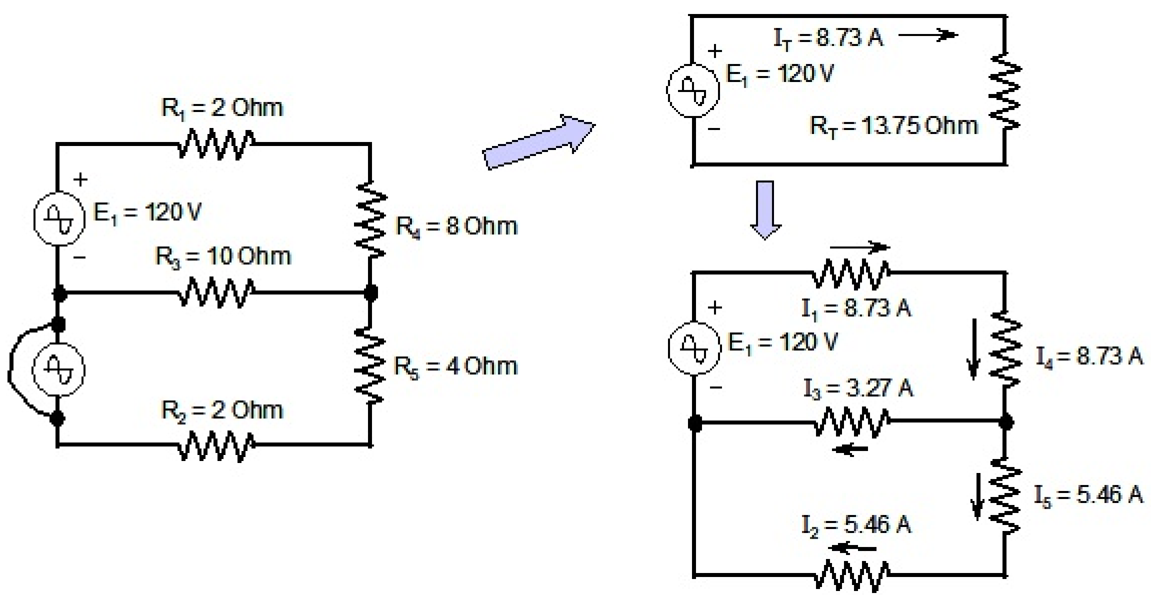

Figure 230.6 *The component currents are determined for the voltage source in the top loop of Figure 230.4 with the voltage source in the lower loop shorted across to eliminate it from the circuit.*

Next repeat the previous process to determine the current through each component of the circuit of Figure 230.4 caused by the voltage source in the lower loop. The currents that flow due to the voltage source in the lower loop are shown in Figure 230.7. Short across the voltage source in the top loop to eliminate it from the circuit. As in the previous step, determine the total resistance (R_T) in order to find the total current. The two loops are not symmetrical, therefore, the total resistance and total current will be different from the previous values. The total resistance (R_T) of this circuit is 11 ohm, and the total current (I_T) is 10.91 amperes. This total current flows through resistors R_2 and R_5, and splits to flow through the parallel branches of the circuit. The current through resistor R_3 is 5.45 amperes, and through resistors R_1 and R_4 is also 5.45 amperes. Note that the two parallel branches have equal resistance resulting in an even split of the total current. The arrows in Figure 230.7 indicate the correct direction of current flow through each component of the circuit.

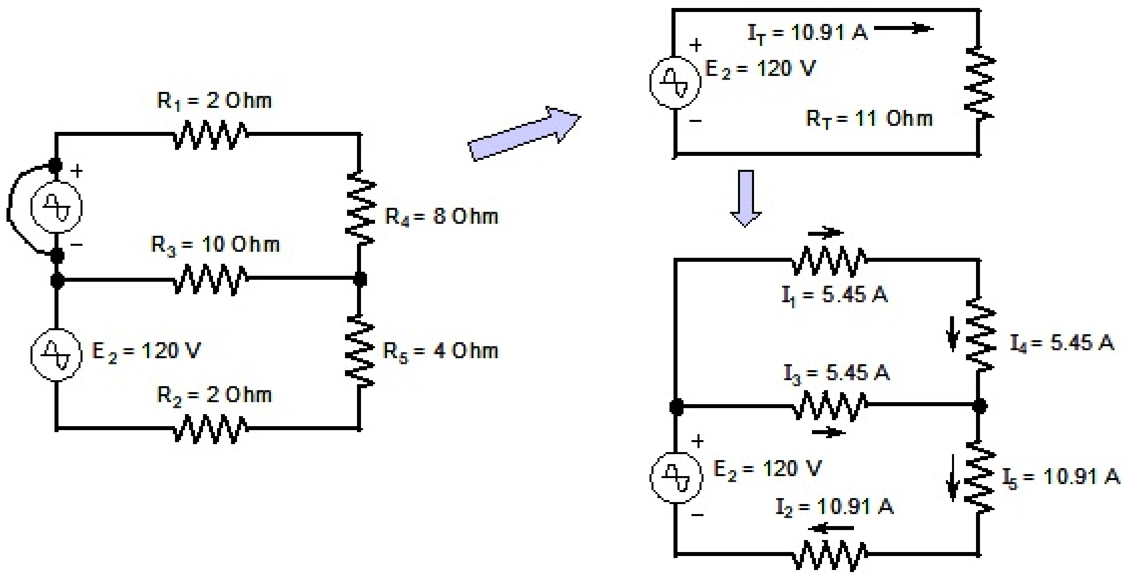

Figure 230.7 *The component currents are determined for the voltage source in the lower loop of Figure 230.4 with the voltage source in the top loop shorted across to eliminate it from the circuit.*

The final step is to superimpose the currents determined for each individual voltage source and determine the net current for each component as shown in Figure 230.8. For the resistor common to both loops (R_3), the currents are in opposite directions and will subtract. For the other components of this circuit the currents will add. The net current for each component of the circuit determined using the Kirchhoff voltage equations as shown in Figure 230.5 are the same as the net current for each component determined using the method of superposition and shown in Figure 230.8.

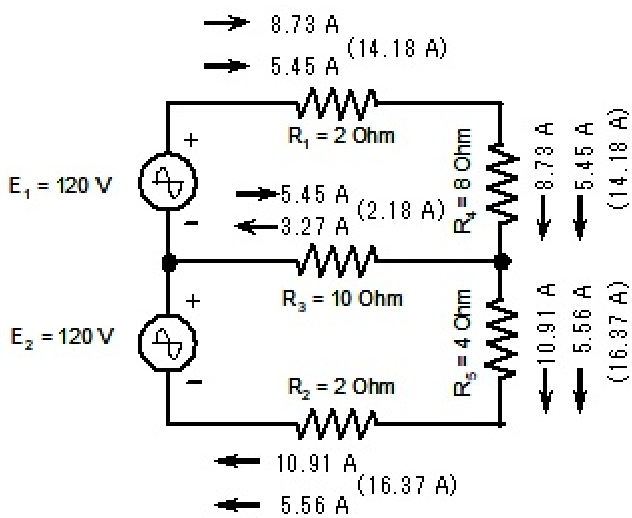

Figure 230.8 *The currents determined for each voltage source and shown in Figure 230.6 and Figure 230.7 are superimposed onto the original circuit of Figure 230.4 and the net current through each component is determined.*

Thevenin Equivalent Circuit: When a circuit analysis only needs to be solved one time, the use of multiple equations following Kirchhoff's laws or the superposition technique are generally the preferred method. When working with real problems in the field, analysis of a circuit multiple times with one component changed is common. Many circuits are often too complex to even solve using these methods, yet analysis of the circuit is necessary. Another analysis technique is to simplify the original circuit into what is known as an equivalent circuit, which from the point of concern acts the same as the original complex circuit. One such circuit is called a Thevenin equivalent circuit where any complex circuit can be reduced to a simple circuit consisting of a voltage source in series with a resistance. An equivalent circuit will be determined that will model the original circuit of Figure 230.4 between points A and B as shown in Figure 230.9.

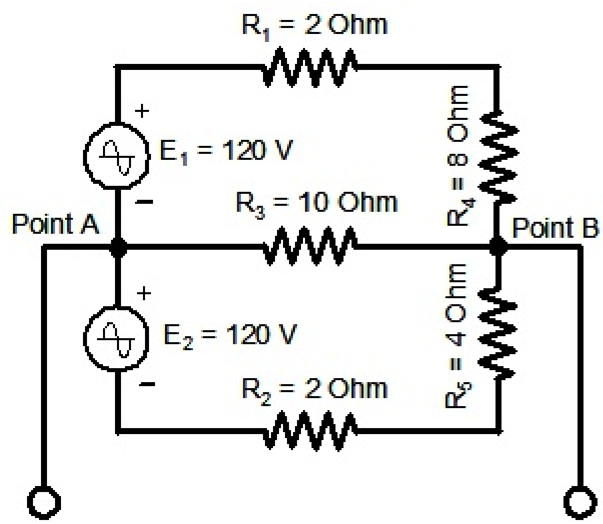

Figure 230.9 *The original circuit of Figure 230.4 will be analyzed across points A and B and will be modeled as a Thevenin equivalent circuit as viewed across the points A and B.*

The circuit of Figure 230.4 is actually a 3-wire, 120/240 volt single-phase supply typical of the power for a dwelling. The common resistor R_3 is the neutral wire between the voltage source and the electrical panel in the building. The net current through the common resistor R_3 was determined earlier to be 2.18 amperes (see Figures 5 and 8). The voltage drop across the 10 ohm resistor R_3 is 21.8 volts. This voltage drop is too high and not acceptable. As shown in Figure 230.10, the circuit will be modeled as a Thevenin equivalent circuit as viewed between points A and B. In this case the open circuit voltage will be determined using the previous analysis of the circuit. The open circuit voltage between points A and B (21.8 volts) is the current through R_3 (2.18 amperes) times the value of R_3 (10 ohm). Usually the value of the open circuit voltage can be measured in the field with a high impedance meter, although in may situations the circuit is dynamic and the open circuit voltage changes over time. This is called the open circuit voltage because the original circuit is undisturbed and no additional resistance has been placed between points A and B that would alter the original circuit. For a static circuit, this is the highest voltage that can exist between points A and B.

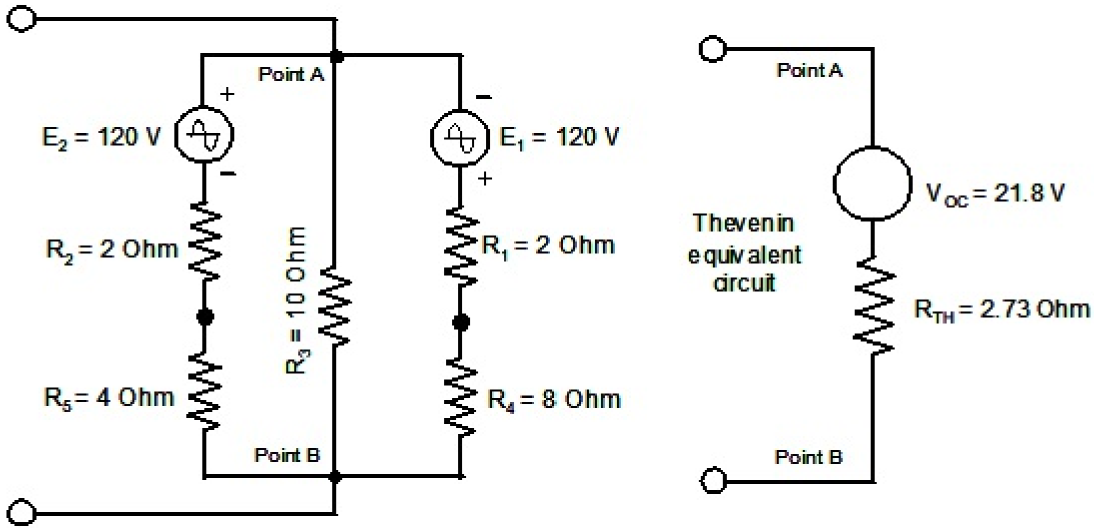

Figure 230.10 *A Thevenin equivalent of a circuit consists of an open circuit voltage (V_{OC}) in series with a resistance called the Thevenin equivalent resistance (R_{TH}) or sometimes the source resistance.*

The Thevenin equivalent resistance (R_{TH}) or source resistance is determined by shorting across all voltage sources in the original circuit and finding the total resistance of the circuit as viewed from points A and B. The original circuit is shown rearranged in the left portion of Figure 230.10 to shown that there are three parallel branches of the circuit between points A and B. With the voltage sources shorted across, the three parallel branches of the circuit have values of 6 ohm, 10 ohm, and 10 ohm. The resistance between points A and B in this case is the Thevenin equivalent resistance (R_{TH}) of the circuit which has a value of 2.73 ohm. The Thevenin equivalent resistance can often be determined experimentally in the field, but it is not necessarily a simple measurement.

Once the Thevenin equivalent circuit has been determined, different values of resistance can be placed across points A and B to determine the effect on the original circuit. What is of interest in this case is the voltage between points A and B, and the current that will flow through the different values of resistors placed between points A and B.

Norton Equivalent Circuit: In some cases it may be desirable to model a circuit so that the current available from two points of a circuit can be studied. This would be the case when a circuit is producing essentially a constant current. A circuit can be modeled as a Norton equivalent between two points such as points A and B of Figure 230.4. The Norton equivalent consists of a constant current source in parallel with a source resistance as shown in Figure 230.11. It may be possible to determine these values experimentally in the field, or they can be determined by analysis from a circuit such as Figure 230.4. It turns out that the Norton equivalent circuit and the Thevenin equivalent circuit are related, and one can easily be determined from the other. As shown in Figure 230.11, short the terminals of the Thevenin equivalent circuit and calculate the current flow (I_{SC} = short circuit current) using Equation 230.1. This is the maximum current that can be delivered by the Thevenin equivalent circuit, and the value of the short circuit current (I_{SC}) is the open circuit voltage (V_{OC}) divided by the source resistance (R_{TH}) using Equation 230.2. The value of the source resistance (R_{TH}) in each circuit is identical. The value of the open circuit voltage (V_{OC}) can be determined from the Norton equivalent by multiplying the short circuit current (I_{SC}) by the source resistance (R_{TH}) using Equation 230.2.

Norton Equivalent: $I_{SC} = \dfrac{V_{OC}}{R_{TH}}$ **Equation 230.1**

Thevenin Equivalent: $V_{OC} = I_{SC} \times R_{TH}$ **Equation 230.2**

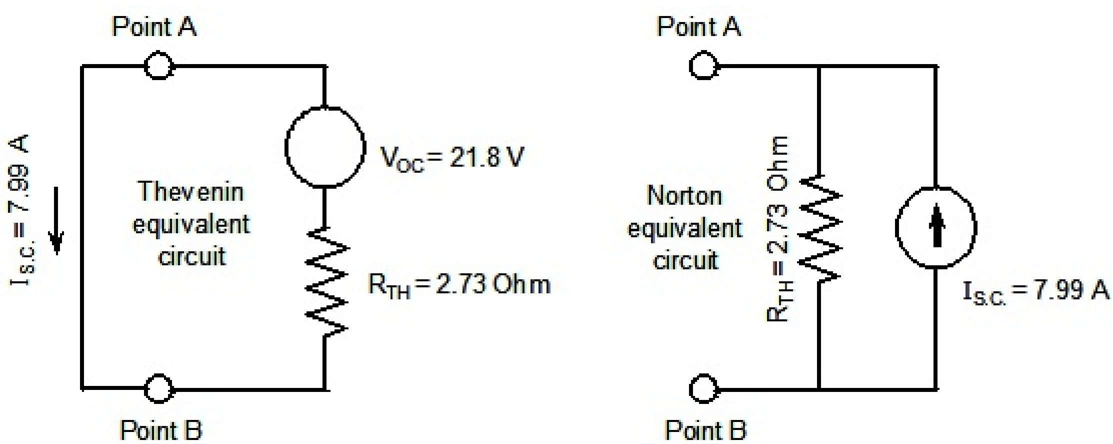

Figure 230.11 *A Norton equivalent of a circuit consists of a constant current source (I_{SC}) in parallel with a source resistance which is the same as the Thevenin equivalent resistance (R_{TH}).*

Conclusions: There are two types of components in an electrical circuit, sources and sinks. A source produces a voltage and the current flow within the source is from negative to positive, but outside the source the current flows from the positive terminal, through the circuit and returns to the negative terminal of the source. A sink, such as a resistor, uses power or converts electrical energy into another form such as heat. Within a sink the current flows from the positive end to the negative end.

Kirchhoff's laws must be satisfied for each circuit or loop within a circuit. The voltages across the components of a loop must add to zero. This means the positive voltages must equal the negative voltages for that loop. This is Kirchhoff's voltage law. The current law states that for any junction point (node) the currents must add to zero. This means that for any junction point the current entering the junction point (node) must equal the current leaving that junction point. After solving for the current and voltage of a circuit, using these two laws can be handy in determining whether an error in calculation has occurred. Kirchhoff's laws can be useful in solving for the currents and voltages for the components of a circuit by writing an equation for each loop of a circuit, or for each node of a circuit.

When a circuit has multiple sources of voltage, it can be useful to solve for a circuit using the principle of superposition. With this technique, the circuit is solved individually for each voltage source and then all solutions are superimposed upon each other to determine the resultant current in each component and voltage across each component. It is extremely important to correctly indicate the direction of current flow for each component in each individual solution. The importance of this technique is that the significance of each source at a point of interest can be determined.

Use of the Thevenin or Norton equivalent circuit is a convenient way to model a circuit or to analyze the effect of different treatments with respect to the operation of that circuit. For example the effect of varying the value of a resistor at one point in a circuit can often be more easily determined by first modeling the circuit as a Thevenin or Norton equivalent circuit, then changing the value of resistor of interest in the analysis. Understanding how to use the basic rules of circuit analysis are powerful tools when analyzing both simple and complex electrical circuits.

The examples in this Tech Note were limited to simple circuits with resistors. These techniques also work for ac circuits with complex elements (complex numbers) consisting of resistance, capacitance, and inductance.