Electrical Tech Note — 222

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RLC Series and Parallel Circuit Analysis

An RLC circuit consists of components that are resistors (R), inductors (L), and capacitors (C) in an alternating current circuit where the voltage source is a sine wave. For the purpose of this discussion, the voltage source will be considered to be 60 Hz and the voltage and current levels will be the rms values. Often circuits made up of these components can be analyzed fairly easily. It may be useful to review *Tech Note 221* which discusses capacitors and inductors in alternating current circuits along with a brief discussion of polar and rectangular representation of voltage, current, and impedance.

Series Connection of a Resistor, Capacitor, and Inductor: A resistor, capacitor, and inductor are shown connected in series for the circuit of Figure 222.1. There is only one path through this series circuit, and there will be only one current. The same current passes through all of the circuit elements. For convenience it will be assumed the supply voltage when represented on a polar (phasor) diagram will have an angle of 0°. In order to determine the current flow in the circuit it is necessary to determine the impedance (Z) of the circuit. Then Ohm's law can be used to determine the current.

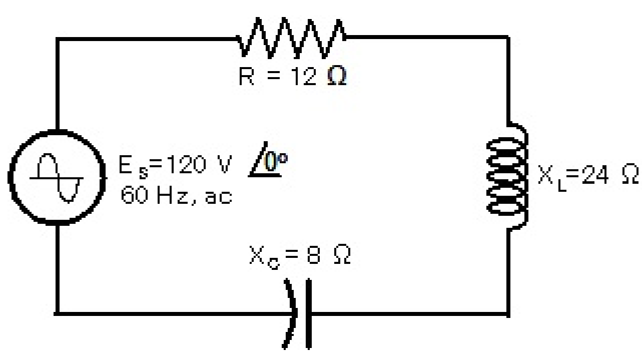

Figure 222.1 A resistor with a value of 12 ohms is connected in series with a capacitor with a capacitive reactance of 8 ohms and an inductor with an inductive reactance of 24 ohms.

The resistance (R), capacitive reactance (X_C), and inductive reactance (X_L) of the circuit of Figure 222.1 are shown plotted on the impedance diagram of Figure 222.2. Inductive reactance has an angle of +90°, capacitive reactance has an angle of −90°, and resistance has an angle of 0°. For a series circuit, resistance, capacitive reactance, and inductive reactance can be added as vectors. It can be seen in Figure 222.2 that the net reactance forms the vertical leg of a right triangle, and the resistance forms the horizontal leg. Impedance will be the hypotenuse of the right triangle and is equal to the square root of the sum of the squares of the two legs of the triangle as represented by Equation 222.1. The value of the impedance for the circuit of Figure 222.1 is worked out on the next page and has a value of 20 ohm.

$$Z = \sqrt{R^2 + (X_C - X_L)^2}$$ **Equation 222.1**

$$Z = \sqrt{12^2 + (24 - 8)^2} = \sqrt{144 + 256} = \sqrt{400} = 20\Omega$$

Impedance can be represented in polar form as a magnitude and an angle, or it can be represented in rectangular form as a horizontal and vertical component. In rectangular form impedance is the sum of the resistance, capacitive reactance, and the inductive reactance as represented by Equation 222.2 which for the example of Figure 222.1 is Z = 12 + j16.

$Z = R + X_C + X_L = R + jX$ **Equation 222.2**

where $j = \sqrt{-1}$

Z = [12 + j0] + [0 – j8] + [0 + j24] = 12 + j16

The impedance can be represented in polar form by determining the magnitude using Equation 222.1, and the angle by a method from trigonometry such as Equation 222.4. Refer to *Tech Note 221* for a brief review of these trigonometry formulas.

$$\theta = \tan^{-1}\frac{X}{R}$$ **Equation 222.4**

magnitude of $Z = \sqrt{12^2 + 16^2} = 20\Omega$

angle of Z is $\theta = \tan^{-1}\frac{16}{12} = 53^\circ$

$Z = 20 \angle{+53^\circ}$ ohm

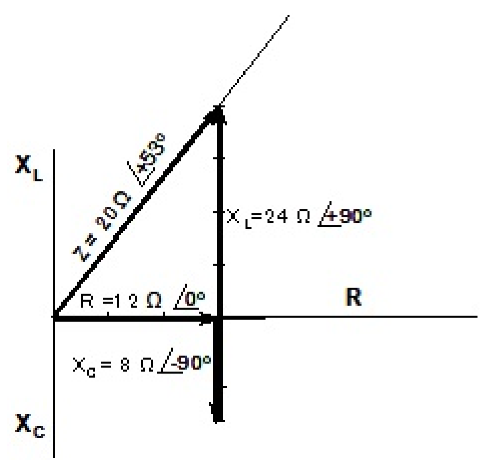

Figure 222.2 For a series circuit the resistance, capacitive reactance, and inductive reactance can be added to form the legs of a right triangle. The impedance will be the hypotenuse of the triangle.

Now that the supply voltage and circuit impedance are known, the circuit current can be determined using Ohm's law as indicated by Equation 222.5, however, impedance is used rather than resistance. The value of the current in polar form for the circuit of Figure 222.1 is 6 amperes with an angle of minus 53°. For a brief review of math in polar form refer to *Tech Note 221*.

$$I = \frac{E}{Z}$$ **Equation 222.5**

$$\text{Current} = \frac{120\ \angle 0^\circ}{20\ \angle{+53^\circ}} = 6\ \angle{-53^\circ}\ \text{A}$$

The voltage drop across each element in a series circuit such as the circuit of Figure 222.1 can be determined by multiplying the circuit current by the resistance (R), capacitive reactance (X_C), or the inductive reactance (X_L). Since this is a series circuit, the voltage drop across each element of the circuit will add to obtain the supply voltage of the circuit. The voltages are shown added in Figure 222.3. The voltage drop across the capacitor can be subtracted from the voltage drop across the inductor to obtain the vertical leg of a right triangle.

$$E_R = I \times R$$ **Equation 222.6**

$$E_R = 6\ \angle{-53^\circ} \times 12\ \angle 0^\circ = 72\ \angle{-53^\circ}\ \text{V}$$

$$E_C = I \times X_C$$ **Equation 222.7**

$$E_C = 6\ \angle{-53^\circ} \times 8\ \angle{-90^\circ} = 48\ \angle{-143^\circ}\ \text{V}$$

$$E_L = I \times X_L$$ **Equation 222.8**

$$E_L = 6\ \angle{-53^\circ} \times 24\ \angle{+90^\circ} = 144\ \angle{+37^\circ}\ \text{V}$$

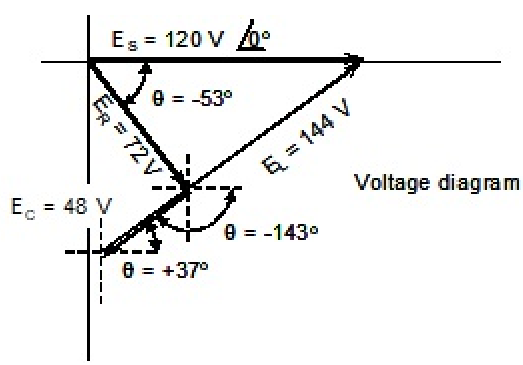

Figure 222.3 For a series circuit, the voltage drop across each element in the circuit can be added as a vector to obtain the supply voltage of the circuit.

If the voltage diagram in Figure 222.3 is rotated counter-clockwise 53°, it is easy to see the diagram forms a right triangle with the circuit supply voltage as the hypotenuse. Equation 222.9 can then be used to determine the magnitude of the supply voltage of the circuit which in the case of the circuit of Figure 222.1 is 120 volts.

$$E_S = \sqrt{E_R^{\,2} + (E_C - E_L)^2} \qquad \textbf{Equation 222.9}$$

$$E_S = \sqrt{72^2 + (48 - 144)^2} = \sqrt{5184 + 9216} = \sqrt{14400} = 120V$$

The previous discussion explained how the supply voltage of the circuit can be determined in polar form. The supply voltage can also be determined in rectangular form by adding the values of voltage drop across each element in the series circuit. If the voltage across an element in a circuit is known in polar form with a magnitude and an angle, it is useful to be able to convert that value to rectangular form. The process involves the use of two functions from trigonometry. With the hypotenuse known, multiply that value by the cosine of the angle to obtain the length of the horizontal (x) component as indicated by Equation 222.10. The vertical component (y) is represented with the imaginary number **j** which can be obtained by multiplying the hypotenuse by the sine of the angle as indicated by Equation 222.11. The voltage across each component of the series circuit of Figure 222.1 is $E_R = 43 - j58$, $E_C = 38 - j29$, and $E_L = 115 + j87$. The voltage drop across each of these components can be added in rectangular form as indicated by Equation 222.12 to obtain the value of the supply voltage $E_S = 120 + j0$.

$$x = \text{hypotenuse} \times \cos\theta \qquad \textbf{Equation 222.10}$$

$$y = j(\text{hypotenuse} \times \sin\theta) \qquad \textbf{Equation 222.11}$$

$$E_R = [72 \times \cos{-53^\circ}] + [j(72 \times \sin{-53^\circ})] = 43 - j58$$

$$E_C = [48 \times \cos{-143^\circ}] + [j(48 \times \sin{-143^\circ})] = -38 - j29$$

$$E_L = [144 \times \cos{+37^\circ}] + [j(144 \times \sin{+37^\circ})] = 115 + j87$$

$$E_S = E_R + E_C + E_L \quad \text{(rectangular form)} \qquad \textbf{Equation 222.12}$$

$$E_S = [43 - j58] + [-38 - j29] = 120 + j0$$

Parallel Connection of a Resistor, Inductor, and Capacitor: When the resistor, capacitor, and inductor are connected in parallel the resistance, capacitive reactance, and inductive reactance cannot be added to obtain the impedance of the circuit. The reciprocal of the circuit impedance is the sum of the reciprocals of the impedances of each parallel branch of the circuit, but this process involves more complex mathematics. When the parallel circuit is simple as shown in Figure 222.4, consisting of resistance in one branch, inductive reactance in another branch, and capacitive reactance in the still another branch, the total circuit current is fairly easy to determine. Use Ohm's law to determine the current in each branch of the parallel circuit, then determine the total current of the circuit using Equation 222.17 as described later. Finally divide the circuit voltage by the total current to obtain the impedance of the parallel circuit as indicated by Equation 222.19. The current through the capacitor will lead the current through the resistor by 90°, and the current through the inductor will lag behind the current through the resistor by 90°. Actually the current through the capacitor and the current through the inductor are out-of-phase by 180°. This means these two currents will subtract from each other. When the supply voltage is known in polar form, and the resistance, capacitive reactance, and inductive reactance are all known in polar form, Equation 222.13, Equation 222.14, and Equation 222.15 can be used to determine the current through each branch of the parallel circuit. For the parallel circuit example of Figure 222.4 the currents are $I_R = 10 \angle 0^\circ$, $I_C = 15 \angle +90^\circ$, and $I_L = 5 \angle -90^\circ$. In this case the current through the capacitor has an angle of +90° and the current through the inductor has an angle of −90° which makes it easy to convert these currents into rectangular form.

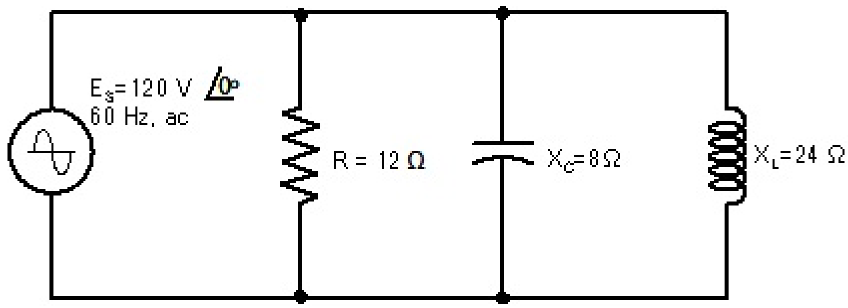

Figure 222.4 A resistor with a value of 12 ohms is connected in parallel with a capacitor with a capacitive reactance of 8 ohms and an inductor with an inductive reactance of 24 ohms.

$$I_R = \frac{E_S}{R}$$ **Equation 222.13**

$$I_R = \frac{E_S}{R} = \frac{120 \angle 0^\circ}{12 \angle 0^\circ} = 10 \angle 0^\circ \text{ A} = 10 + j0$$

$$I_C = \frac{E_S}{X_C}$$ **Equation 222.14**

$$I_C = \frac{E_S}{X_C} = \frac{120 \angle 0^\circ}{8 \angle -90^\circ} = 15 \angle +90^\circ \text{ A} = 0 + j15$$

$$I_L = \frac{E_S}{X_L}$$ **Equation 222.15**

$$I_L = \frac{E_S}{X_L} = \frac{120 \angle 0^\circ}{24 \angle +90^\circ} = 5 \angle -90^\circ \text{ A} = 0 + j5$$

The easiest method of determining the impedance of a circuit with a resistor, capacitor, and an inductor in parallel is to first determine the total current flowing in the circuit, then use Ohm's law to find the impedance (Z), Equation 222.19. If the currents through the individual elements are known in rectangular form, they can be added to determine the total circuit current using Equation 222.16, which in the case of the circuit of Figure 222.4 is $I_T = 10 + j10$. The individual parallel branch currents are shown added together in Figure 222.5. Note that the current through the capacitor and the current through the inductor subtract to form the vertical leg ($I_C - I_L$) of a right triangle, and the current through the resistor forms the horizontal leg (I_R). The length of the hypotenuse is the magnitude of the total circuit current (I_T). For this circuit the voltage in polar from is assumed to have an angle of 0°, therefore, the angle at which the current is leading or lagging the voltage can be determined by taking the inverse tangent of the two sides of the right triangle as indicated by Equation 222.18. The magnitude of the total current can be determined by taking the square root of the square of the current through the resistor plus the square of the difference of the current through the capacitor and the inductor using Equation 222.17. For this circuit the total current is 14.1 ∠+45° amperes.

$$I_T = I_R + I_C + I_L$$ (rectangular form) **Equation 222.16**

$$I_T = [10 - j0] + [0 + j15] + [0 - j5] = 10 + j10$$

$$I_T = \sqrt{I_R^{\,2} + (I_C - I_L)^2}$$ (magnitude of current) **Equation 222.17**

$$I_T = \sqrt{10^2 + (15 - 5)^2} = \sqrt{100 + 100} = \sqrt{200} = 14.1A$$

$$\theta = \tan^{-1} \frac{(I_C - I_L)}{I_R}$$ (angle of total current) **Equation 222.18**

$$\theta = \tan^{-1} \frac{(15 - 5)}{10} = 45^\circ$$

The impedance of the circuit of Figure 222.4 with the resistor, capacitor, and inductor connected in parallel is determined by dividing the supply voltage by the total current using Equation 222.19. For the parallel circuit example of Figure 222.4 the impedance of the circuit is 8.5 ohm with an angle of minus 45º.

$$Z = \frac{E_S}{I_T}$$ **Equation 222.19**

$$Z = \frac{120 \angle 0^\circ}{14.1 \angle +45^\circ} = 8.5 \angle -45^\circ \text{ Ohm}$$

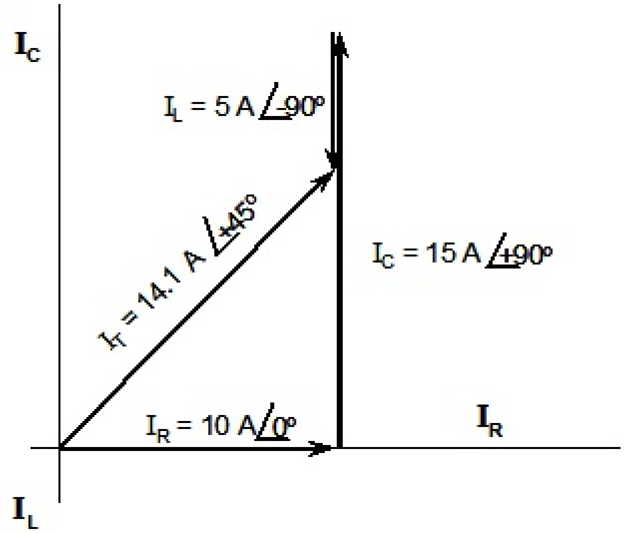

Figure 222.5 When connected in parallel, the currents through the resistor, capacitor, and inductor can be added with the net current through the capacitor and inductor forming the vertical leg of a right triangle on the current diagram. The total circuit current will be the hypotenuse of the triangle.

Current in an RLC Circuit: The process of building an inductor involves a length of wire usually wound around a laminated steel core. The length of wire will have a significant resistance which means it is generally not possible to separate the resistance from the inductive reactance. Figure 222.6 has one branch with a resistor in series with an inductor. To offset the inductance of the circuit it is common to connect a capacitor in parallel with the RL branch of the circuit. By using the previously discussed rules for solving series and parallel circuits, a solution for a circuit such as shown in Figure 6 can be determined. The resistor and inductor are in series, therefore, solve for the impedance of the RL branch of the circuit. The 9 ohm resistance is the horizontal side of the right triangle, and the 12 ohm inductive reactance if the vertical side of the right triangle. The impedance of that branch of the circuit is the hypotenuse of the right triangle. The impedance angle of the RL branch is determined by taking the inverse tangent of the inductive reactance (12 ohm) divided by the resistance (9 ohm).

$$Z_{RL} = 9 + j12 = 15 \angle +53^\circ$$

$$Z_{RL} = \sqrt{9^2 + 12^2} = \sqrt{225} = 15 ohm$$

$$\theta = \tan^{-1} \frac{12}{9} = +53^\circ$$

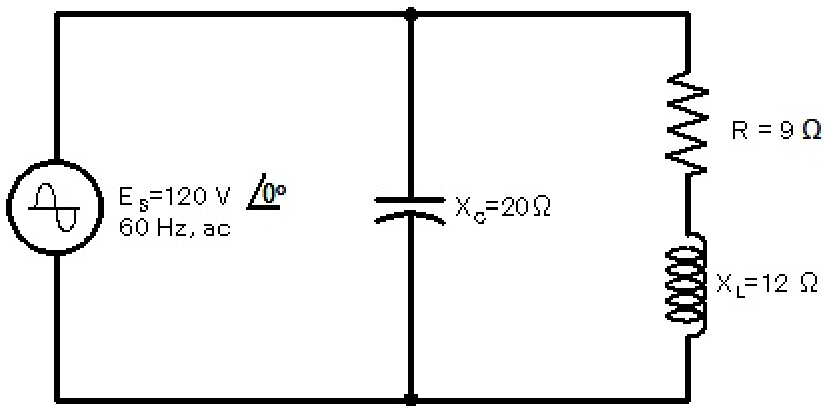

Figure 222.6 A resistor with a value of 9 ohms is connected in series with an inductor that has an inductive reactance of 12 ohms. Connected in parallel with the RL branch is a capacitor with a capacitive reactance of 20 ohms.

Once the impedance of the RL branch of the circuit is known, the current through that branch can be determined using Ohm's law. The current through the capacitor can also be determined using Ohm's law. The currents are shown added as vectors in Figure 222.7. The currents can be added in any order, however, for Figure 7 the current through the RL branch was plotted first and the current through the capacitor plotted second. The total current of the circuit can be determined directly by adding the current through the RL branch of the circuit in rectangular form to the current through the capacitor in rectangular form as indicated by Equation 222.20 to obtain a total current of (4.81 – j0.39) ampere. The total current can then be plotted on the diagram without having to resort to plotting the current through each branch. Once the total current is known in rectangular form, the magnitude of the total current can be determine using Equation 222.17 which in the case of the circuit of Figure 222.6 is 4.83 $\angle -5^o$. Note one of the benefits of adding a capacitor in parallel with an RL circuit by observing the magnitude of the total current in comparison with the current through the capacitor and the RL branch. The total current is less then either the current through the capacitor or the current through the RL branch of the circuit. This technique is called power factor correction, and it has as its purpose to reduce the total current drawn by an RL device.

$$I_{RL} = \frac{E_S}{Z_{RL}} = \frac{120 \angle 0^\circ}{15 \angle +53^\circ} = 8 \angle -53^\circ \text{ A} = 4.81 - j6.39$$

$$I_C = \frac{E_S}{X_C} = \frac{120 \angle 0^\circ}{20 \angle -90^\circ} = 6 \angle +90^\circ \text{ A} = 0 + j6.00$$

$$I_T = I_{RL} + I_C$$ (rectangular form) **Equation 222.20**

$$I_T = [4.81 - j6.39] + [0 + j6.00] = 4.81 - j0.39$$

$$I_T = \sqrt{4.81^2 + (-0.39)^2} = 4.83A$$

$$\theta = \tan^{-1} \frac{0.39}{4.81} = -5^\circ$$

$I_T = 4.83 \angle -5^\circ$ A

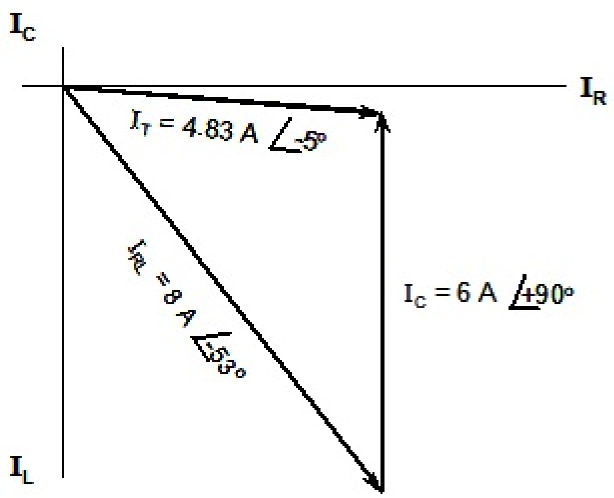

Figure 222.7 By installing a capacitor in parallel with an RL circuit, the total current of the circuit can be less than the current through the capacitor or the current through the RL branch of the circuit.